ON THE MATRIX $A \otimes A^*$ OF SIZE 2×2 IN MAX-PLUS ALGEBRA

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Abstract: Max-plus algebra is an analogue of linear algebra, developed for a pair of operations denoted as $\oplus$ and $\otimes$, which we call max-algebraic addition and multiplication, respectively. These operations, when extended to vectors and matrices, redefine the classical approach to linear algebraic operations. Specifically, replacing addition and multiplication with max-plus operations results in a novel mathematical framework that enables nonlinear problems to be analyzed and solved in a linear manner. We often call max-plus algebra simply max-algebra (we can also talk about min-plus algebra, or max-time algebra, but those structures are not the subject of this paper). This mathematical area began its development in the 1960s, and since then, a substantial body of research has been published. Many of these papers address topics central to classical linear algebra, such as eigenvalues and eigenvectors, solving systems of equations, matrix ranks, and matrix powers. Max-algebra has also found practical applications in fields like manufacturing, transportation, traffic light control, resource allocation, and the natural sciences. However, one notable limitation in max-algebra is the nonexistence of inverse matrices in the general case - they are only defined for generalized permutation matrices. This complicates problem-solving, as inverse matrices are indispensable in classical linear algebra for tasks like solving systems of equations. Consequently, new tools and methodologies must be developed to address foundational issues within max-algebra. One such tool, beside dual operations, max-algebraic permanent, and maximum cycle mean, is the conjugate matrix, defined as the negative transpose matrix of a given matrix. The conjugate matrix has appeared in pioneering works on max-algebra. Also, matrices derived from the max-algebraic sum or product of a matrix and its conjugate exhibit unique properties. For instance, a matrix resulting from the max-algebraic product of a given matrix and its conjugate matrix is a square matrix with a zero diagonal. With this in mind, this paper examines the following question: given a square matrix with a zero diagonal, is it possible to determine a matrix A such that the max-algebraic product of A and its conjugate matrix equals the given matrix? Due to the complexity of this problem, we focus on the simplest case: when a square matrix of dimensions 2×2 is given. This serves as a basis for further exploration and the derivation of new insights regarding matrices of higher dimensions. A square matrix of dimensions 2×2 with a zero diagonal inherently contains two off-diagonal elements. Consequently, as we shall demonstrate, solving the problem requires examining two distinct cases. These cases are explained through theorem along with a proof. An example illustrating the proven facts is also given in the paper.
Keywords: max-plus algebra, conjugate matrix, 2×2 matrix, zero diagonal

1. INTRODUCTION

Max-algebra, also referred to as max-plus algebra, is a mathematical framework that reimagines the principles of linear algebra through the introduction of max-algebraic addition ($\oplus$) and max-algebraic multiplication ($\otimes$). These operations, defined as $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$, for all $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, extend naturally to vectors and matrices. The resulting structure $(\overline{\mathbb{R}}, \oplus, \otimes)$ forms a commutative and idempotent semiring, providing a novel perspective for analyzing and solving nonlinear problems using linear-like operations.
The origins of max-algebra date back to the 1960s with foundational contributions from researchers such as Cuninghame-Green, Carre, Gondran and Minoux, and Vorobyov, among others. Since then, max-algebra has gained prominence due to its practical applications across various fields, including discrete event systems, scheduling, synchronization, traffic control, and transportation. These applications stem from the ability of max-algebra to model problems involving optimization and decision-making in systems where "maximum" and "addition" are natural operators.
Max-algebra also introduces unique challenges and mathematical concepts. Many classical problems from linear algebra, such as finding eigenvalues and eigenvectors, solving systems of equations, and determining the rank or power of matrices, have been reinterpreted in the context of max-algebra. However, one significant limitation is the absence of inverse matrices in the general case, which restricts the use of certain classical methods. To overcome these challenges, new tools and methodologies have been developed, including the use of conjugate matrices, which are defined as the negative transpose of a given matrix.
This paper focuses on a specific problem within the realm of max-algebra: given a square matrix $\tilde{A}$ with a zero diagonal, is it possible to find a matrix A such that the max-algebraic product of A and its conjugate equals the given

matrix, i.e., such that $\tilde{A} = A \otimes A^*$? The product of matrices A and A^* is a square matrix with a zero diagonal. Therefore, we assume that such a matrix is given; otherwise, it would not be possible to consider this question.
Given the complexity of this problem, we first examine the simplest case: a 2×2 square matrix. This foundational analysis lays the groundwork for further exploration of higher-dimensional cases and opens up new avenues for research in max-algebra.
The remainder of this paper is organized as follows. In Section 2, we present the basic concepts and properties of max-algebra that will be utilized in subsequent sections. The main result is provided in Section 3, highlighted by Theorem 1, which is accompanied by a detailed proof. This is followed by an example illustrating the theorem's statement, along with a discussion of its significance and potential directions for future research. A brief conclusion is provided at the end of the paper.

2. MATERIALS AND METHODS

Here we outline the fundamental definitions and properties of max-plus algebra that form the basis for proving the primary claims of this paper. For more detailed information, we refer the readers to Butkovič (2010).
If, within a set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, we substitute the standard operations of addition $(+)$ and multiplication $(\times)$ from classical linear algebra with newly defined operations $(\oplus, \otimes)$, as

$$a \oplus b = max\{a, b\}$$

and

$$a \otimes b = a + b,$$

this yields the structure $(\overline{\mathbb{R}}, \oplus, \otimes)$, referred to as *max-plus algebra* (or simply *max-algebra*). This structure is recognized as a semiring, which is commutative and idempotent.
In max-algebra, the neutral element for the operation $\oplus$ is $-\infty$, while the neutral element for the operation $\otimes$ is 0. Therefore, the identity matrix in max-algebra is a square matrix where diagonal elements are 0 and off-diagonal elements are $-\infty$. The definition of a transpose of the matrix in max-algebra aligns with classical linear algebra, maintaining the same principles. Additionally, max-algebra introduces the concept of a conjugate matrix. Since there are no inverse matrices in max-algebra (in the general case), the conjugate matrix partially replaces the inverse matrix.
Definition 1: Let a matrix A be given. The *conjugate matrix*, denoted A^*, is the negative transpose of the matrix A:

$$A^* = -A^T.$$

Max-algebraic addition, multiplication and scalar multiplication, for matrices $A, B, C \in \overline{\mathbb{R}}$ of compatible sizes, are defined as follows:

$$C = A \oplus B \quad if \quad c_{ij} = a_{ij} \oplus b_{ij}, \qquad for\ all\ i, j;$$

$$C = A \otimes B \quad if \quad c_{ij} = \sum_{k}^{\oplus} a_{ik} \otimes b_{kj} = \max_{k}\{a_{ik} + b_{kj}\} \ , \qquad for\ all\ i, j;$$

$$\tau \otimes A = A \otimes \tau = (\tau \otimes a_{ij}), \qquad for\ all\ \tau \in \mathbb{R}.$$

Also, throughout this paper, we will denote by $N = \{1,2,3, \dots, n\}$, where $n \in \mathbb{N}$.
In max-algebra, the absence of subtraction introduces the need for dual operations corresponding to $(\oplus, \otimes)$. These operations, denoted as $(\oplus', \otimes')$, are defined as follows:

$$a \oplus' b = min\{a, b\}$$

and

$$a \otimes' b = a + b,$$

for all $a, b \in \overline{\overline{\mathbb{R}}} = \mathbb{R} \cup \{+\infty\}$.
We extend these operations to matrices and vectors in the same way as $(\oplus, \otimes)$.
In this case, the additive identity is $+\infty$, forming the structure referred to as *min-plus algebra (*or simply *min-algebra)*. Remarkably, all statements valid in max-algebra hold true in their corresponding dual form within min-algebra. This involves replacing concepts such as "maximum" with "minimum" and "positive" with "negative", etc.

Definition 2: A *max-algebraic pseudo skew-symmetric matrix* (*MAPSS matrix*) is a square matrix with a zero diagonal that does not contain two symmetric negative elements.
It is straightforward to show that matrix $\tilde{A} = A \otimes A^*$ belongs to the class of MAPSS matrices. Other information related to MAPSS matrices can be found in Stojčetović (2025).

3. RESULTS AND DISCUSSIONS

Due to the unique definitions of operations in max-algebra, the properties of matrices $A \oplus A^*$ and $A \otimes A^*$ differ from their counterparts in classical linear algebra. However, rather than addressing these properties, we focus on the matrix $A \otimes A^*$ within the following context.

First, denote $\tilde{A} = A \otimes A^*$. We will consider the following question: if a square matrix $\tilde{A}$ with a zero diagonal is given, is it possible, and if so, how, to determine the matrix A such that $A \otimes A^*$ equals the given matrix? This question naturally extends to the problem of solving max-linear systems of equations.

Here, we will describe the case of matrices from □^{2x2}.

Theorem 1: Let $\tilde{A} \in \mathbb{R}^{2x2}$ be a given matrix with a zero diagonal. The matrix A such that $A \otimes A^* = \tilde{A}$ is any matrix whose off-diagonal elements satisfy one of the following two conditions:

$$a_{11} - a_{21} = \tilde{a}_{12} \quad \text{and} \quad a_{12} - a_{22} = -\tilde{a}_{21},$$

or

$$a_{12} - a_{22} = \tilde{a}_{12} \quad \text{and} \quad a_{11} - a_{21} = -\tilde{a}_{21}.$$

Proof:

Let $\tilde{A} \in \mathbb{R}^{2x2}$ be a given matrix with a zero diagonal:

$$\tilde{A} = \begin{bmatrix} 0 & \tilde{a}_{12} \\ \tilde{a}_{21} & 0 \end{bmatrix}.$$

Taking into account how the max-algebraic multiplication of matrices is defined, we have:

$$A \otimes A^* = \tilde{A} \quad \Leftrightarrow \quad \max_{i,j}\{a_{ij} + a_{ji}^*\} = \tilde{a}_{ij}, \quad i,j \in N.$$

Since $N = \{1,2\}$, for the elements of matrix $\tilde{A}$, the following must hold:

$$\max\{a_{11} + a_{11}^*, a_{12} + a_{21}^*\} = 0, \tag{1.1}$$
$$\max\{a_{11} + a_{12}^*, a_{12} + a_{22}^*\} = \tilde{a}_{12}, \tag{1.2}$$
$$\max\{a_{21} + a_{11}^*, a_{22} + a_{21}^*\} = \tilde{a}_{21}, \tag{1.3}$$
$$\max\{a_{21} + a_{12}^*, a_{22} + a_{22}^*\} = 0. \tag{1.4}$$

Since (1.1) and (1.4) surely hold (because $a_{ij}^* = a_{ji}$), we are left to consider (1.2) and (1.3). These equations can also be expressed as:

$$\max\{a_{11} - a_{21}, a_{12} - a_{22}\} = \tilde{a}_{12},$$
$$\max\{a_{21} - a_{11}, a_{22} - a_{12}\} = \tilde{a}_{21}.$$

Let us introduce the following notations:

$$a_{11} - a_{21} = p \quad \text{and} \quad a_{12} - a_{22} = q.$$

Now, we have:

$$\max\{p, q\} = \tilde{a}_{12},$$
$$\max\{-p, -q\} = \tilde{a}_{21}.$$

From here, there are two cases we can consider.

1. $\max\{p, q\} = p \quad \Rightarrow \quad \max\{-p, -q\} = -q,$

which implies $p = \tilde{a}_{12}$ and $-q = \tilde{a}_{21}$, i.e.:

$$a_{11} - a_{21} = \tilde{a}_{12} \quad \text{and} \quad a_{12} - a_{22} = -\tilde{a}_{21}.$$

2. $\max\{p, q\} = q \quad \Rightarrow \quad \max\{-p, -q\} = -p,$

and thus $q = \tilde{a}_{12}$ and $-p = \tilde{a}_{21}$, i.e.:

$$a_{12} - a_{22} = \tilde{a}_{12} \quad \text{and} \quad a_{11} - a_{21} = -\tilde{a}_{21}.$$

Therefore, the theorem is proven. ■

Let's also note that Theorem 1 will hold in the case $p = q$, which is easy to conclude.

Corollary 1: The matrix A satisfying $A \otimes A^* = \tilde{A}$, where $\tilde{A}$ is a given matrix with a zero diagonal, is not unique. ■

Example 1: Let the matrix $\tilde{A}$ be given:

$$\tilde{A} = \begin{bmatrix} 0 & 13 \\ -4 & 0 \end{bmatrix}.$$

Then, based on the Theorem 1, for the elements of the matrix A, one of the following conditions must hold:

$$a_{11} - a_{21} = 13 \quad \text{and} \quad a_{12} - a_{22} = 4,$$

or

$$a_{12} - a_{22} = 13 \quad \text{and} \quad a_{11} - a_{21} = 4.$$

We can choose, for example, $a_{11} = 16$, $a_{21} = 3$, $a_{12} = 8$, and $a_{22} = 4$. Thus, we get the matrix:

$$A = \begin{bmatrix} 16 & 8 \\ 3 & 4 \end{bmatrix}.$$

For this matrix, it holds:

$$A \otimes A^* = \begin{bmatrix} 16 & 8 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} -16 & -3 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 13 \\ -4 & 0 \end{bmatrix} = \tilde{A}.$$

Another solution to this problem is, for example, the matrix B:

$$B = \begin{bmatrix} 6 & 17 \\ 2 & 4 \end{bmatrix}.$$

This is easy to verify, as the matrix elements satisfy the second condition from Theorem 1. □

Note that in this example, taking into account that $\tilde{A}$ is a MAPSS matrix, we assumed that the symmetric elements are not both negative.

However, there exist matrices with a zero diagonal that are not the max-algebraic product of a matrix and its conjugate matrix. Thus, even if we apply the conditions described in Theorem 1, we cannot obtain a matrix A such that $A \otimes A^*$ is equal to the given matrix with a zero diagonal. This can be easily verified by a counterexample.

The conjugate matrix plays a crucial role in max-algebra, as the inverse matrix is not defined in the general case. Consequently, the conjugate matrix serves as a substitute for the inverse matrix to some extent. It is among the key tools used to solve one-sided and two-sided systems in max-algebra. The problem addressed in this study can be categorized within this class, as it involves determining unknown elements of a matrix. Specifically, the focus here has been on the simplest case, which concerns 2×2 matrices. Nonetheless, this basic case provides a solid foundation for further investigation in this domain.

It is significant to describe the properties of the matrix $A \otimes A^*$ (i.e. of $A^* \otimes A$), because this matrix can be important in solving two-sided max-algebraic systems of equations. Specifically, for solving two-sided systems we can use the so-called Alternating method, an iterative procedure that starts with a randomly chosen vector. Under certain conditions that should apply to the matrices of that system, it is possible to determine an upper bound of the number of iterations at the Alternating method. Namely, if A is a finite integer matrix of that system of dimension $n \times n$, then the number of iterations at the Alternating method is bounded by:

$$(n-1)(1 + {x_0}^* \otimes A^* \otimes A \otimes x_0),$$

where x_0 is the initial random vector.

It is also important to mention that everything said above is valid in min-algebraic theory, of course, in dual form. The matrix $A \otimes' A^*$ is also a square matrix with a zero diagonal, so bearing in mind the cases we have considered here, we can draw conclusions about the dual claim in min-algebra.

4. CONCLUSIONS

In this paper, we have presented how to determine a matrix A such that the max-algebraic product of A and its conjugate matrix yields a given 2×2 matrix with a zero diagonal. This approach can be extended to matrices of higher dimensions. However, for 3×3 matrices, the complexity of the problem increases significantly due to the larger number of cases to be considered. Nevertheless, by starting with smaller matrices, valuable insights can be gained that, in the ideal scenario, could lead to solving the problem for the general case, i.e., for $n \times n$ matrices.

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